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Lipschitz stability in an inverse problem for the Korteweg-de Vries equation on a finite domain

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Abstract

In this paper, we address an inverse problem for the Korteweg-de Vries equation posed on a bounded domain with boundary conditions proposed by Colin and Ghidaglia. More precisely, we retrieve the principal coefficient from the measurements of the solution on a part of the boundary and also at some positive time in the whole space domain. The Lipschitz stability of this inverse problem relies on a Carleman estimate for the linearized Korteweg-de Vries equation and the Bukhgeĭm-Klibanov method.

MSC: 35R30; 35Q53

Keywords: inverse problem; Korteweg-de Vries equation; Carleman estimate

1 Introduction

This paper is concerned with the Korteweg-de Vries (KdV) equation with a non-constant coefficient posed on a finite interval

$$\begin{cases} y_t + a(x)y_{xxx} + y_x + yy_x = 0, & \text{in } Q, \\ y(0,t) = y_x(L,t) = y_{xx}(L,t) = 0, & \text{in } (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,L), \end{cases} \tag{1.1}$$

where $L, T > 0$, $Q = (0,L) \times (0,T)$, the initial data y_0 is known and the unknown coefficient $a = a(x)$ is assumed to be time independent.

The KdV equation,

$$y_t + y_x + y_{xxx} + yy_x = 0,$$

was first derived by Korteweg and de Vries [1] in 1895 (or by Boussinesq [2] in 1876) as a model for the propagation of some surface water waves along a channel. In applications to physical problems, the independent variable x is often a coordinate representing position in the medium of propagation, t is proportional to elapsed time, and $y(x,t)$ is a velocity or an amplitude at point x at time t. Based on the results in [3–6], if $h = h(x)$ is the function describing the variations in depth of the channel, then the KdV equation becomes (after scaling)

$$y_t + (\sqrt{h}y)_x + h^2 y_{xxx} + \frac{1}{\sqrt{h}} y y_x = 0.$$

Later, in [7], the main coefficient $h^2 y_{xxx}$ was corrected by $h^{\frac{5}{2}} y_{xxx}$. Therefore, it is meaningful to consider the inverse problem of retrieving the principal coefficient in the KdV equation.

The KdV equation on a finite domain has been extensively studied in the past. Most of this work has been focused on the following system:

$$\begin{cases} y_t + y_{xxx} + y_x + y y_x = 0, & \text{in } Q, \\ y(0,t) = y(L,t) = y_x(L,t) = 0, & \text{in } (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,L), \end{cases} \tag{1.2}$$

which possesses a different set of boundary conditions than those of system (1.1). In recent years, system (1.1) ($a \equiv 1$) has attracted many authors' attention. The well-posedness, controllability and stabilization of (1.1) ($a \equiv 1$) have been studied in [8–13].

In this paper we intend to retrieve the principal coefficient $a = a(x)$ of system (1.1) from the measurement of $y(L,t)$ on $(0,T)$ and the measurement of $y(x,T/2)$ on $(0,L)$. Stability estimates play a special role in the theory of inverse problems of mathematical physics that are ill-posed in the classical sense. They determine the choice of regularization parameters and the rate at which solutions of regularized problems converge to an exact solution. The results concerning the determination of coefficients for parabolic equations and hyperbolic equations are relatively rich (see [14–16] and the references therein). Concerning a dispersive equation, the results focused on the Schrödinger equation ([17–19]). However, to the best of our knowledge, the only result in the literature concerning the determination of coefficients for the KdV equation is in [20], where the author considered the KdV equation with boundary conditions as in (1.2).

Let us introduce the following notations for functional spaces appearing in this paper:

$$X^s = C\big([0,T], H^s(0,L)\big) \cap L^2\big(0,T; H^{s+1}(0,L)\big),$$
$$Y = \big\{ y \in X^6 \mid y_t \in X^3 \text{ and } y_{tt} \in X^0 \big\}.$$

To precisely state the results in this article, we introduce the sets

$$\Sigma(a_0,\alpha) = \big\{ a \in W^{3,\infty}(0,L) \mid \forall x \in (0,L), a(x) \geq a_0 > 0 \text{ and } \|a\|_{W^{3,\infty}(0,L)} \leq \alpha \big\},$$
$$W = \big\{ w \in H^6(0,L) \mid w(0) = w'(L) = w''(L) = 0 \big\}.$$

The following theorem is the main result of this article.

Theorem 1.1 *Let a_0, α and K be given positive constants, $y_0 \in W$ and $a, \tilde{a} \in \Sigma(a_0,\alpha)$. Assume that there exists $\eta > 0$ such that* $\inf\{|\tilde{y}_{xxx}(x, \frac{T}{2})|, x \in (0,L)\} \geq \eta$. *Then there exists a* *positive constant* $C = C(T, \alpha, \eta, \alpha_0, K, L)$ *such that*

$$C\|a - \tilde{a}\|_{L^2(0,L)} \leq \|y(L,t) - \tilde{y}(L,t)\|_{H^1(0,T)} + \left\| y\left(\cdot, \frac{T}{2}\right) - \tilde{y}\left(\cdot, \frac{T}{2}\right)\right\|_{H^3(0,L)}$$

for all y *and* $\tilde{y}$ *satisfying* $\max\{\|y\|_Y, \|\tilde{y}\|_Y\} \leq K$.

Remark 1.1 Compared with [20], Theorem 1.1, the symmetry hypothesis on the initial data is replaced by a condition on the trajectory.

Remark 1.2 To prove Theorem 1.1, we need to establish a Carleman estimate for a linearized KdV equation with boundary conditions in (1.2). To the best of our knowledge, it is the first attempt to establish the Carleman estimate for a third order operator with these boundary conditions.

The rest of this paper is organized as follows. Section 2 is devoted to a Carleman estimate for the linearized KdV equation with non-constant main coefficient. In Section 3, we prove Theorem 1.1 following the Bukhgeĭm-Klibanov method.

2 Carleman estimate

In this section, we provide the suitable Carleman estimate for the study of the stability of our inverse problem.

Let us consider the operator

$$P = \partial_t + a(x)\partial_{xxx} + b(x,t)\partial_x + d(x,t)$$

defined on

$$\mathcal{V} = \{v \in L^2(0,T;H^3(0,L)) \mid v(0,t) = v_x(L,t) = v_{xx}(L,t) = 0, t \in (0,T), \\ \text{and } Pv \in L^2(Q)\}.$$

Here $b \in L^\infty(0,T;W^{1,\infty}(0,L))$, $d \in L^\infty(Q)$ and $a \in \Sigma(a_0, \alpha)$.

Consider $\beta \in C^3([0,L])$ such that for some $r > 0$ we have

$$0 < r \leq \beta(x) \quad \text{and} \quad 0 < r \leq \beta'(x), \quad \forall x \in (0,L).$$

We define, for each $\lambda > 0$, the functions

$$\phi(x,t) = \frac{e^{2\lambda\|\beta\|_\infty} - e^{\lambda\beta(x)}}{t(T-t)}, \qquad \theta(x,t) = \frac{e^{\lambda\beta(x)}}{t(T-t)},$$

for $(x,t) \in Q$. It is not difficult to see that θ satisfies the following properties:

$$\exists C > 0 \quad \text{such that } \frac{1}{C}\theta \leq \theta_x \leq C\theta \quad \text{and}$$
$$\theta^n \leq C\theta^m \quad \text{for each positive integers } n < m.$$

Theorem 2.1 *Let ϕ, θ and P be defined as above. There exist $s_0 > 0$, $\lambda_0 > 0$ and a constant $C = C(T, s_0, \lambda_0, a_0, \alpha, L) > 0$ such that, for every $s \geq s_0$, $\lambda \geq \lambda_0$,*

$$\begin{aligned}&\int_Q e^{-2s\phi}\left(s^5\lambda^6\theta^5u^2 + s^3\lambda^4\theta^3u_x^2 + s\lambda^2\theta u_{xx}^2\right)dx\,dt \\ &\quad\leq C\int_Q e^{-2s\phi}|Pu|^2\,dx\,dt + C\int_0^T \left(s^5\lambda^5\theta^5e^{-2s\phi}u^2\right)\big|_{x=L}\,dt.\end{aligned} \tag{2.1}$$

Proof Following the method in [21], it is enough to prove (2.1) for $\widetilde{P} = \partial_t + a(x)\partial_{xxx}$. In fact, assume that we have proved (2.1) for $\tilde{P}$, we have

$$\int_Q e^{-2s\phi}|\widetilde{P}u|^2\,dx\,dt \leq C\left(\int_Q e^{-2s\phi}|Pu|^2\,dx\,dt + \int_Q e^{-2s\phi}\left(u_x^2 + u^2\right)dx\,dt\right).$$

By choosing $s > 0$ and $\lambda > 0$ large, it is possible to absorb $\int_Q e^{-2s\phi}(u_x^2 + u^2)\,dx\,dt$ with the left-hand side of (2.1), concluding that (2.1) also holds for P.

Let $s > 0$ and consider the operator P_ϕ defined in $\mathcal{W}_s = \{e^{-s\phi}u \mid u \in \mathcal{V}\}$ by

$$P_\phi w = e^{-s\phi}\widetilde{P}\left(e^{s\phi}w\right).$$

We then obtain the decomposition $P_\phi w = P_1w + P_2w + Rw$, where

$$\begin{aligned}&P_1w = w_t + 3as^2\phi_x^2w_x + aw_{xxx} + 3as^2\phi_x\phi_{xx}w, \\ &P_2w = as^3\phi_x^3w + 3as\phi_xw_{xx} + 3s(a\phi_x)_xw_x, \\ &Rw = as\phi_{xxx}w + 3as^2\phi_x\phi_{xx}w + s\phi_tw - 3sa_x\phi_xw_x - 3as^2\phi_x\phi_{xx}w.\end{aligned} \tag{2.2}$$

Thus

$$\|P_\phi w - Rw\|_{L^2(Q)}^2 = \|P_1w\|_{L^2(Q)}^2 + 2\langle P_1w, P_2w\rangle_{L^2(Q)} + \|P_2w\|_{L^2(Q)}^2.$$

Let us now consider $\langle P_1w, P_2w\rangle_{L^2(Q)}$.

Claim that

$$\langle P_1w, P_2w\rangle_{L^2(Q)} = \int_Q\left((\cdot)w^2 + (\cdot)w_x^2 + (\cdot)w_{xx}^2\right)dx\,dt + \int_0^T (\cdot)\,\big|_{x=0}^{x=L}\,dt, \tag{2.3}$$

where

$$\begin{aligned}(\cdot)w^2 = &\left(-\frac{1}{2}s^3a\left(\phi_x^3\right)_t - \frac{3}{2}s^5\left(a^2\phi_x^5\right)_x - \frac{1}{2}s^3\left(a^2\phi_x^3\right)_{xxx} + 3s^5a^2\phi_x^4\phi_{xx}\right. \\ &\left.+ \frac{9}{2}s^3\left(a^2\phi_x^2\phi_{xx}\right)_{xx} - \frac{9}{2}s^3\left(a\phi_x\phi_{xx}(a\phi_x)_x\right)_x\right)w^2, \\ (\cdot)w_x^2 = &\left(\frac{3}{2}sa\phi_{xt} - \frac{9}{2}s^3\left(a^2\phi_x^3\right)_x + 9s^3a\phi_x^2(a\phi_x)_x + \frac{3}{2}s^3\left(a^2\phi_x^3\right)_x\right. \\ &\left.+ \frac{3}{2}s\left(a(a\phi_x)_x\right)_{xx} - 9s^3a^2\phi_x^2\phi_{xx}\right)w_x^2,\end{aligned}$$

$$
\begin{aligned}
(\cdot)w_{xx}^2 &= \left(-\frac{3}{2}s\bigl(a^2\phi_x\bigr)_x - 3sa(a\phi_x)_x\right)w_{xx}^2, \\
(\cdot)\big|_{x=0}^{x=L} &= \biggl(3sa\phi_x w_t w_x + \frac{3}{2}s^5a^2\phi_x^5w^2 + \frac{9}{2}s^3a^2\phi_x^3w_x^2 + s^3a^2\phi_x^3ww_{xx} \\
&\quad - s^3\bigl(a^2\phi_x^3\bigr)_x ww_x - \frac{1}{2}s^3a^2\phi_x^3w_x^2 + \frac{1}{2}s^3\bigl(a^2\phi_x^3\bigr)_{xx}w^2 + \frac{3}{2}sa^2\phi_x w_{xx}^2 \\
&\quad - \frac{3}{2}s\bigl(a(a\phi_x)_x\bigr)_x w_x^2 + 3sa(a\phi_x)_x w_x w_{xx} + 9s^3a^2\phi_x^2\phi_{xx}ww_x \\
&\quad - \frac{9}{2}s^3\bigl(a^2\phi_x^2\phi_{xx}\bigr)_x w^2 + \frac{9}{2}s^3a\phi_x\phi_{xx}(a\phi_x)_x w^2\biggr)\bigg|_{x=0}^{x=L}.
\end{aligned}
$$

Indeed, (2.3) follows easily from the following equations:

$$
\begin{aligned}
&\int_Q as^3\phi_x^3ww_t\,dx\,dt = -\int_Q \frac{1}{2}s^3a\bigl(\phi_x^3\bigr)_t w^2\,dx\,dt, \\
&\int_Q 3as\phi_x w_t w_{xx}\,dx\,dt = \int_0^T (3sa\phi_x w_t w_x)\big|_{x=0}^{x=L}\,dt \\
&\qquad\qquad + \int_Q \left(\frac{3}{2}sa\phi_{xt}w_x^2 - 3s(a\phi_x)_x w_t w_x\right)dx\,dt, \\
&\int_Q 3a^2s^5\phi_x^5ww_x\,dx\,dt = \int_0^T \left(\frac{3}{2}s^5a^2\phi_x^5w^2\right)\bigg|_{x=0}^{x=L}\,dt - \int_Q \frac{3}{2}s^5\bigl(a^2\phi_x^5\bigr)_x w^2\,dx\,dt, \\
&\int_Q 9s^3a^2\phi_x^3w_x w_{xx}\,dx\,dt = \int_0^T \left(\frac{9}{2}s^3a^2\phi_x^3w_x^2\right)\bigg|_{x=0}^{x=L}\,dt - \int_Q \frac{9}{2}s^3\bigl(a^2\phi_x^3\bigr)_x w_x^2\,dx\,dt, \\
&\int_Q s^3a^2\phi_x^3ww_{xxx}\,dx\,dt = \int_Q \left(\frac{3}{2}s^3\bigl(a^2\phi_x^3\bigr)_x w_x^2 - \frac{1}{2}s^3\bigl(a^2\phi_x^3\bigr)_{xxx}w^2\right)dx\,dt \\
&\qquad\qquad + \int_0^T s^3\biggl(a^2\phi_x^3ww_{xx} - \bigl(a^2\phi_x^3\bigr)_x ww_x - \frac{1}{2}a^2\phi_x^3w_x^2 \\
&\qquad\qquad + \frac{1}{2}\bigl(a^2\phi_x^3\bigr)_{xx}w^2\biggr)\bigg|_{x=0}^{x=L}\,dt, \\
&\int_Q 3sa^2\phi_x w_{xx}w_{xxx}\,dx\,dt = \int_0^T \left(\frac{3}{2}sa^2\phi_x w_{xx}^2\right)\bigg|_{x=0}^{x=L}\,dt - \int_Q \frac{3}{2}s\bigl(a^2\phi_x\bigr)_x w_{xx}^2\,dx\,dt, \\
&\int_Q 3sa(a\phi_x)_x w_x w_{xxx}\,dx\,dt = \int_Q \left(\frac{3}{2}s\bigl(a(a\phi_x)_x\bigr)_{xx}w_x^2 - 3sa(a\phi_x)_x w_{xx}^2\right)dx\,dt \\
&\qquad\qquad + \int_0^T \left(-\frac{3}{2}s\bigl(a(a\phi_x)_x\bigr)_x w_x^2 + 3sa(a\phi_x)_x w_x w_{xx}\right)\bigg|_{x=0}^{x=L}\,dt, \\
&\int_Q 9s^3a^2\phi_x^2\phi_{xx}ww_{xx}\,dx\,dt = \int_Q \left(\frac{9}{2}s^3\bigl(a^2\phi_x^2\phi_{xx}\bigr)_{xx}w^2 - 9s^3a^2\phi_x^2\phi_{xx}w_x^2\right)dx\,dt \\
&\qquad\qquad + \int_0^T \left(9s^3a^2\phi_x\phi_{xx}ww_x - \frac{9}{2}s^3\bigl(a^2\phi_x^2\phi_{xx}\bigr)_x w^2\right)\bigg|_{x=0}^{x=L}\,dt, \\
&\int_Q 9s^3a\phi_x\phi_{xx}(a\phi_x)_x ww_x\,dx\,dt = -\int_Q \frac{9}{2}s^3\bigl(a\phi_x\phi_{xx}(a\phi_x)_x\bigr)_x w^2\,dx\,dt \\
&\qquad\qquad + \int_0^T \left(\frac{9}{2}s^3a\phi_x\phi_{xx}(a\phi_x)_x w^2\right)\bigg|_{x=0}^{x=L}\,dt.
\end{aligned}
$$

Similar to [20], we obtain

$$\begin{aligned}&\int_Q \big((\cdot)w^2 + (\cdot)w_x^2 + (\cdot)w_{xx}^2\big)\,dx\,dt \\ &\quad= \int_Q \left(\frac{9}{2}s^5\lambda^6 a^2\beta_x^6\theta^5 w^2 + 9s^3\lambda^4 a^2\beta_x^4\theta^3 w_x^2 + \frac{9}{2}s\lambda^2 a^2\beta_x^2\theta w_{xx}^2\right) dx\,dt + I_R, \qquad (2.4)\end{aligned}$$

where I_R gathers the non-dominating terms and satisfies the requirement that, for any $\varepsilon > 0$, there exist $s_0 > 0$, $\lambda_0 > 0$ such that if $s \geq s_0$, $\lambda \geq \lambda_0$, we have

$$|I_R| \leq \varepsilon \int_Q \big(s^5\lambda^6\theta^5 w^2 + s^3\lambda^4\theta^3 w_x^2 + s\lambda^2\theta w_{xx}^2\big)\,dx\,dt. \qquad (2.5)$$

Then we consider the boundary terms. Write

$$\begin{aligned}\int_0^T (\cdot)\,\Big|_{x=0}^{x=L}\,dt = \int_0^T \bigg(&-\frac{3}{2}s^5\lambda^5 a^2\beta_x^5\theta^5 w^2 - 8s^3\lambda^3 a^2\beta_x^3\theta^3 w_x^2 - \frac{3}{2}s\lambda a^2\beta_x\theta w_{xx}^2 \\ &- 3as\lambda\theta\beta_x w_t w_x - a^2 s^3\lambda^3\beta_x^3\theta_x^3 w w_{xx} + B_R\bigg)\bigg|_{x=0}^{x=L}\,dt, \qquad (2.6)\end{aligned}$$

where B_R can be estimated as follows:

$$\begin{aligned}|B_R| = \bigg|&-s^3\big(a^2\phi_x^3\big)_x w w_x + \frac{1}{2}s^3\big(a^2\phi_x^3\big)_{xx} w^2 \\ &- \frac{3}{2}s\big(a(a\phi_x)_x\big)_x w_x^2 + 3sa(a\phi_x)_x w_x w_{xx} \\ &+ 9s^3 a^2\phi_x^2\phi_{xx} w w_x - \frac{9}{2}s^3\big(a^2\phi_x^2\phi_{xx}\big)_x w^2 + \frac{9}{2}s^3 a\phi_x\phi_{xx}(a\phi_x)_x w^2\bigg| \\ \leq{}& C\big(s^3\lambda^4\theta^3|w||w_x| + s^3\lambda^5\theta^3 w^2 + s\lambda^3\theta w_x^2 + s\lambda^2\theta|w_x||w_{xx}| \\ &+ s^3\lambda^4\theta^3|w||w_x| + s^3\lambda^4\theta^2 w^2 + s^3\lambda^5\theta^3 w^2\big).\end{aligned}$$

For any $\varepsilon > 0$, there exist $s_0 > 0$, $\lambda_0 > 0$ such that if $s \geq s_0$, $\lambda \geq \lambda_0$, we have

$$|B_R| \leq \varepsilon\big(s^5\lambda^5\theta^5 w^2 + s^3\lambda^3\theta^3 w_x^2 + s\lambda\theta w_{xx}^2\big).$$

Now we estimate each term of the right-hand side in (2.6).

It is easy to deduce that

$$\begin{aligned}w_x &= -s\phi_x e^{-s\phi}u + e^{-s\phi}u_x, \\ w_t &= -s\phi_t e^{-s\phi}u + e^{-s\phi}u_t, \\ w_{xx} &= -s\phi_{xx}e^{-s\phi}u + s^2\phi_x^2 e^{-s\phi}u - 2s\phi_x e^{-s\phi}u_x + e^{-s\phi}u_{xx}.\end{aligned}$$

Noting that $u(0,t) = u_t(0,t) = u_x(L,t) = u_{xx}(L,t) = 0$, $t \in (0,T)$, we have

$$\begin{aligned}
&w(0,t) = 0, \qquad w(L,t) = e^{-s\phi(L,t)}u(L,t),\\
&w_t(0,t) = 0, \qquad w_t(L,t) = e^{-s\phi(L,t)}u_t(L,t) - s\phi_t(L,t)e^{-s\phi(L,t)}u(L,t),\\
&w_x(0,t) = e^{-s\phi(0,t)}u_x(0,t),\\
&w_x(L,t) = s\lambda\beta_x(L)\theta(L,t)e^{-s\phi(L,t)}u(L,t),\\
&w_{xx}(0,t) = 2s\lambda\beta_x(0)\theta(0,t)e^{-s\phi(0,t)}u_x(0,t) + e^{-s\phi(0,t)}u_{xx}(0,t),\\
&w_{xx}(L,t) = s\lambda\bigl(\beta_{xx}(L) + \lambda\beta_x^2(L) + s\lambda\beta_x^2(L)\theta(L,t)\bigr)\theta(L,t)e^{-s\phi(L,t)}u(L,t).
\end{aligned} \tag{2.7}$$

Since $\beta_x \geq r > 0$ and $\theta > 0$, we can obtain from (2.7)

$$\begin{aligned}
&\frac{3}{2}s^5\lambda^5a^2\beta_x^5\theta^5w^2 + 8s^3\lambda^3a^2\beta_x^3\theta^3w_x^2 + \frac{3}{2}s\lambda a^2\beta_x\theta w_{xx}^2\bigg|_{x=0} \geq 0,\\
&\left|\int_0^T (-3as\lambda\beta_x\theta w_tw_x)\,|_{x=0}\,dt\right| = 0,\\
&\left|\int_0^T \bigl(-a^2s^3\lambda^3\beta_x^3\theta^3ww_{xx}\bigr)\,\big|_{x=0}\,dt\right| = 0.
\end{aligned}$$

Taking (2.7) into account and choosing s and λ large enough, we obtain

$$\begin{aligned}
&\int_0^T \left(-\frac{3}{2}s^5\lambda^5a^2\beta_x^5\theta^5w^2 - 8s^3\lambda^3a^2\beta_x^3\theta^3w_x^2 - \frac{3}{2}s\lambda a^2\beta_x\theta w_{xx}^2\right)\bigg|_{x=0}^{x=L}dt\\
&\quad\geq -C\int_0^T \bigl(s^5\lambda^5\theta^5w^2 + s^3\lambda^3\theta^3w_x^2 + s\lambda\theta w_{xx}^2\bigr)\,|_{x=L}\,dt\\
&\quad\geq -C\int_0^T \bigl(s^5\lambda^5\theta^5e^{-2s\phi}u^2\bigr)\,|_{x=L}\,dt,\\
&\left|\int_0^T (-3as\lambda\beta_x\theta w_tw_x)\,|_{x=0}^{x=L}\,dt\right|\\
&\quad= \left|3as\lambda\int_0^T (\beta_x\theta w_tw_x)\,|_{x=L}\,dt\right|\\
&\quad= \left|\int_0^T \bigl(3as^2\lambda^2\beta_x^2\theta^2e^{-2s\phi}u_tu - 3as^3\lambda^2\beta_x^2\theta^2\phi_te^{-2s\phi}u^2\bigr)\,\big|_{x=L}\,dt\right|\\
&\quad= \left|\int_0^T \left(-\frac{3}{2}as^2\lambda^2\bigl(\beta_x^2\theta^2\bigr)_te^{-2s\phi}u^2 - 3as^3\lambda^2\beta_x^2\theta^2\phi_te^{-2s\phi}u^2\right)\bigg|_{x=L}\,dt\right|\\
&\quad\leq C\int_0^T \bigl(s^2\lambda^2\theta^3e^{-2s\phi}u^2 + s^3\lambda^2\theta^4e^{-2s\phi}u^2\bigr)\,\big|_{x=L}\,dt \quad \bigl(\text{since } |\phi_t| + |\theta_t| \leq C\theta^2\bigr)\\
&\quad\leq C\int_0^T \bigl(s^5\lambda^5\theta^5e^{-2s\phi}u^2\bigr)\,\big|_{x=L}\,dt,\\
&\left|\int_0^T \bigl(-a^2s^3\lambda^3\beta_x^3\theta^3ww_{xx}\bigr)\,\big|_{x=0}^{x=L}\,dt\right|\\
&\quad= \left|a^2s^3\lambda^3\int_0^T \bigl(\beta_x^3\theta^3ww_{xx}\bigr)\,\big|_{x=L}\,dt\right|
\end{aligned}$$

$$= \left| a^2 s^3 \lambda^3 \int_0^T \left(\beta_x^3 \theta^3 \left(s\lambda \beta_{xx} \theta + s\lambda^2 \beta_x^2 \theta + s^2 \lambda^2 \beta_x^2 \theta^2 \right) e^{-2s\phi} u^2 \right) \Big|_{x=L} \, dt \right|$$
$$\leq C \int_0^T \left(s^5 \lambda^5 \theta^5 e^{-2s\phi} u^2 \right) |_{x=L} \, dt.$$

Therefore

$$\int_0^T (\cdot) \,|_{x=0}^{x=L} \, dt \geq -C \int_0^T \left(s^5 \lambda^5 \theta^5 e^{-2s\phi} u^2 \right) \Big|_{x=L} \, dt.$$

Combining (2.3)-(2.5), consequently,

$$\int_Q \left(s^5 \lambda^6 \theta^5 w^2 + s^3 \lambda^4 \theta^3 w_x^2 + s\lambda^2 \theta w_{xx}^2 \right) dx\,dt + \int_Q \left(|P_1 w|^2 + |P_2 w|^2 \right) dx\,dt$$
$$\leq C \int_Q |P_\phi w|^2 \, dx\,dt + C \int_0^T \left(s^5 \lambda^5 \theta^5 e^{-2s\phi} u^2 \right) \Big|_{x=L} \, dt. \tag{2.8}$$

Returning w to $e^{-s\phi} u$, we can obtain (2.1). □

3 Inverse problem

We first state the well-posedness results for the KdV equation considered in this paper.

Following the methods developed in [20] with minor changes, we have

Theorem 3.1 *Let a_0, α be given positive constants, $a \in \Sigma(a_0, \alpha)$ and $y_0 \in W$. Then system* (1.1) *has a unique solution in Y.*

Next, the local stability of the nonlinear inverse problem stated in Theorem 1.1 will be proved following the ideas in [22]. For the sake of clarity, we divided the proof in several steps.

Step 1. *Local study of the inverse problem.* Let a, $\tilde{a}$, y and $\tilde{y}$ be defined as in Theorem 1.1. Define $u(x,t) := y(x,t) - \tilde{y}(x,t)$ and $\sigma(x) := \tilde{a}(x) - a(x)$. Then u solves the following equation:

$$\begin{cases} u_t + a(x) u_{xxx} + (1 + \tilde{y}) u_x + y_x u = \sigma(x) \tilde{y}_{xxx}, & \text{in } Q, \\ u(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & \text{in } (0,T), \\ u(x,0) = 0, & \text{in } (0,L). \end{cases} \tag{3.1}$$

In order to obtain an estimate of σ in terms of $u(L,\cdot)$ and $u(\cdot, \frac{T}{2})$, we derive equation (3.1) with respect to time. Thus, $v(x,t) := u_t(x,t)$ satisfies the following equation:

$$\begin{cases} v_t + a(x) v_{xxx} + (1 + \tilde{y}) v_x + y_x v = f, & \text{in } Q, \\ v(0,t) = v_x(L,t) = v_{xx}(L,t) = 0, & \text{in } (0,T), \\ v(x,0) = \sigma(x) y_0'''(x), & \text{in } (0,L), \end{cases}$$

where $f = \sigma(x) \tilde{y}_{xxxt} - y_{xt} u - \tilde{y}_t u_x$.

Step 2. *First use of the Carleman estimate.* Similarly to the proof of the Carleman estimate, we set $w = e^{-s\phi} v$. Then we work on the term

$$I := \int_0^{\frac{T}{2}} \int_0^L w P_1 w \, dx\,dt,$$

where P_1 has been defined in (2.2).

On the one hand,

$$I \leq \frac{1}{2}\int_0^{\frac{T}{2}}\int_0^L w^2\,dx\,dt + \frac{1}{2}\int_0^{\frac{T}{2}}\int_0^L |P_1 w|^2\,dx\,dt. \tag{3.2}$$

On the other hand,

$$\begin{aligned} I &= \int_0^{\frac{T}{2}}\int_0^L w\bigl(w_t + 3as^2\phi_x^2 w_x + aw_{xxx} + 3as^2\phi_x\phi_{xx}w\bigr)\,dx\,dt \\ &= \frac{1}{2}\int_0^L \left|w\left(x,\frac{T}{2}\right)\right|^2 dx + R, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} R &= \int_0^{\frac{T}{2}}\int_0^L \bigl(3as^2\phi_x^2 ww_x + aww_{xxx} + 3as^2\phi_x\phi_{xx}w^2\bigr)\,dx\,dt \\ &= \int_0^{\frac{T}{2}}\int_0^L \left(-3s^2 a_x\phi_x^2 w^2 - \frac{1}{2}a_{xxx}w^2 + a_x w_x^2 - aw_x w_{xx}\right)dx\,dt \\ &\quad + \int_0^{\frac{T}{2}} \left(\frac{3}{2}as^2\phi_x w^2 + aww_{xx} - a_x ww_x + \frac{1}{2}a_{xx}w^2\right)\bigg|_{x=0}^{x=L} dt. \end{aligned}$$

Applying (2.7), we obtain

$$\begin{aligned} &\int_0^{\frac{T}{2}} \left(\frac{3}{2}as^2\phi_x w^2 + aww_{xx} - a_x ww_x + \frac{1}{2}a_{xx}w^2\right)\bigg|_{x=0}^{x=L} dt \\ &\quad = \int_0^{\frac{T}{2}} \left(\left(\frac{3}{2}as^2\lambda\beta_x\theta + as\lambda\bigl(\beta_{xx} + \lambda\beta_x^2\bigr)\theta + as^2\lambda^2\beta_x^2\theta^2 \right.\right. \\ &\qquad \left.\left. - a_x s\lambda\beta_x\theta + \frac{1}{2}a_{xx}\right)e^{-2s\phi}v^2\right)\bigg|_{x=L} dt. \end{aligned}$$

For s and λ large enough, it follows immediately that

$$\begin{aligned} s^{\frac{1}{2}}R &\geq -C\int_0^{\frac{T}{2}}\int_0^L \bigl(s^5\lambda^5\theta^5 w^2 + s^3\lambda^3\theta^3 w_x^2 + s\lambda\theta w_{xx}^2\bigr)\,dx\,dt \\ &\quad + C\int_0^{\frac{T}{2}} \bigl(s^5\lambda^5\theta^5 e^{-2s\phi}v^2\bigr)\big|_{x=L}\,dt \\ &\geq -C\int_0^T\int_0^L \bigl(s^5\lambda^5\theta^5 w^2 + s^3\lambda^3\theta^3 w_x^2 + s\lambda\theta w_{xx}^2\bigr)\,dx\,dt. \end{aligned} \tag{3.4}$$

Combining (3.2)-(3.4) and the Carleman estimate (2.8), we conclude that

$$\begin{aligned} &s^{\frac{1}{2}}\int_0^L \left|w\left(x,\frac{T}{2}\right)\right|^2 dx \\ &\quad \leq s\int_0^{\frac{T}{2}}\int_0^L w^2\,dx\,dt + \int_0^{\frac{T}{2}}\int_0^L |P_1 w|^2\,dx\,dt - 2s^{\frac{1}{2}}R \end{aligned}$$

$$\le C\int_Q \big(s^5\lambda^5\theta^5 w^2 + s^3\lambda^3\theta^3 w_x^2 + s\lambda\theta w_{xx}^2\big)\,dx\,dt + \int_Q |P_1 w|^2\,dx\,dt$$
$$\le C\int_Q e^{-2s\phi} f^2\,dx\,dt + C\int_0^T \big(s^5\lambda^5\theta^5 e^{-2s\phi}v^2\big)\big|_{x=L}\,dt, \tag{3.5}$$

where we use the fact that $1+\tilde{y}\in L^\infty(0,T;W^{1,\infty}(0,L))$ and $y_x\in L^\infty(Q)$.

Noting that $w = e^{-s\phi}v = e^{-s\phi}u_t$, it follows from (3.1) that

$$s^{\frac{1}{2}}\int_0^L \left|w\left(x,\frac{T}{2}\right)\right|^2 dx$$
$$= s^{\frac{1}{2}}\int_0^L \big(e^{-2s\phi}\big|\sigma\tilde{y}_{xxx} - au_{xxx} - (1+\tilde{y})u_x - y_x u\big|^2\big)\big|_{t=\frac{T}{2}}\,dx$$
$$\ge s^{\frac{1}{2}}\int_0^L e^{-2s\phi(x,\frac{T}{2})}\big|\sigma(x)\big|^2\left|\tilde{y}_{xxx}\left(x,\frac{T}{2}\right)\right|^2 dx - Cs^{\frac{1}{2}}\left\|u\left(\cdot,\frac{T}{2}\right)\right\|^2_{H^3(0,L)}$$
$$\ge s^{\frac{1}{2}}\eta^2\int_0^L e^{-2s\phi(x,\frac{T}{2})}\big|\sigma(x)\big|^2 dx - Cs^{\frac{1}{2}}\left\|u\left(\cdot,\frac{T}{2}\right)\right\|^2_{H^3(0,L)}. \tag{3.6}$$

Step 3. *Second use of the Carleman estimate.* Considering

$$f = \sigma\tilde{y}_{xxxt} - y_{xt}u - \tilde{y}_t u_x,$$

$\tilde{y}_{xxxt}\in L^2(0,T;H^1(0,L))$, $y_{xt}\in L^\infty(Q)$ and $\tilde{y}_t\in L^\infty(Q)$, we have

$$\int_Q e^{-2s\phi}f^2\,dx\,dt = \int_Q e^{-2s\phi}|\sigma\tilde{y}_{xxxt} - y_{xt}u - \tilde{y}_t u_x|^2\,dx\,dt$$
$$\le C\int_0^L e^{-2s\phi(x,\frac{T}{2})}\big|\sigma(x)\big|^2\int_0^T \big|\tilde{y}_{xxxt}(x,t)\big|^2\,dt\,dx$$
$$+ C\int_Q e^{-2s\phi}|y_{xt}u + \tilde{y}_t u_x|^2\,dx\,dt$$
$$\le C\int_0^L e^{-2s\phi(x,\frac{T}{2})}\big|\sigma(x)\big|^2 dx + C\int_Q e^{-2s\phi}\big(u^2+u_x^2\big)\,dx\,dt. \tag{3.7}$$

Then we can apply the Carleman estimate (2.1) to (3.1),

$$\int_Q e^{-2s\phi}\big(u^2+u_x^2\big)\,dx\,dt$$
$$\le C\int_Q e^{-2s\phi}|\sigma\tilde{y}_{xxx}|^2\,dx\,dt + C\int_0^T \big(s^5\lambda^5\theta^5 e^{-2s\phi}u^2\big)\big|_{x=L}\,dt$$
$$\le C\int_0^L e^{-2s\phi(x,\frac{T}{2})}\big|\sigma(x)\big|^2 dx + C\int_0^T \big(s^5\lambda^5\theta^5 e^{-2s\phi}u^2\big)\big|_{x=L}\,dt. \tag{3.8}$$

From (3.5)-(3.8), choosing s large enough, we deduce that

$$s^{\frac{1}{2}}\int_0^L e^{-2s\phi(x,\frac{T}{2})}\big|\sigma(x)\big|^2 dx$$
$$\le C\int_0^T \big(s^5\lambda^5\theta^5 e^{-2s\phi}\big(u^2+v^2\big)\big)\big|_{x=L}\,dt + Cs^{\frac{1}{2}}\left\|u\left(\cdot,\frac{T}{2}\right)\right\|^2_{H^3(0,L)}. \tag{3.9}$$

Taking into account that $v = u_t = \partial_t(y - \tilde{y})$, the result of Theorem 1.1 directly follows from (3.9).

Competing interests
The author declares that there are no competing interests regarding the publication of this paper.

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